Conceptual Metaphor Games:
An Embodied Approach to Mathematics Pedagogy

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Abstract

In the theory of embodied cognition, mathematical activity is thought to be facilitated in large part by epistemic "mappings" that translate bodily intuitions about the world into abstract conceptual domains. Since their popularization, these ideas have influenced discourse in mathematics education. Another currently popular topic in mathematics education is the growing evidence that learning games can enhance learning outcomes. Combining these facts, this work proposes to construct real-world or virtual games which directly map onto mathematical concepts via conceptual metaphors. Games of this kind, which I dub *conceptual metaphor games*, intend to encourage learning as exploration, making for an approach to inclusive pedagogy which is both person-centric and theoretically sound such that it can be accessibly presented as separate from the underlying theory. I argue that play and discussion of these games can serve as "grounding" experiences with mathematics that confer understanding through a lighthearted medium. An example is given involving the embodied source domain of frogs hopping on lily pads, demonstrating an implementation of such a conceptual metaphor game in a pedagogical context. Additionally, I implement a demo level of a virtual browser-based game which can extend to a larger project for empirical evaluation.

Contents

Emerging theories in cognition suggest that mathematics is *embodied*. What this means is that mathematical activity arises fundamentally from bodily intuitions individuals use to represent and navigate their world. In the case of humans, this includes the use of pattern-preserving *conceptual metaphors* (Lakoff and Núñez, 2000) between embodied domains and abstract, symbolic representations, after which the symbols can be manipulated by themselves, and then "mapped" back into the original domain to obtain physical results. For example, a human may recognize that they do not have enough beads for a bracelet by first interpreting the physical quantity of beads possessed as a symbolic entity (e.g. "there are seven beads"), then examining some facts in the symbolic realm (e.g., "$7 < 10$"), and finally bringing this back into the "real" ("there are too few beads"). Although the story gets more complicated as metaphors develop between abstract domains and even among metaphors themselves, there is a good case that even logic itself is rooted in temporal and spatial intuitions about the world, as to be argued further in the coming sections.

In addition, emerging theories in cognition suggest that mathematics is *situated*. What this means is that mathematical knowledge is dependent on the environment in which individuals act; that is, embodied intuitions and metaphors reside in their context. The concept of "environment" affords multiple levels of analysis; for example, the environment is both ecological and sociocultural. In the case of the former, the content of mathematics is dependent on the physical properties of the environment, often perceived *amodally*, i.e., echoing across the senses (as in: two sounds, two sights, two tastes, etc.). The latter, however, is just as important; knowledge is embedded within *communities of practice*, wherein learning becomes an innately social process by which learners not only become acquainted with adequate tools for interpretation of content but also form cultural identities by participation in the community (Lave and Wenger, 1991). For this reason, human mathematics is not "culture-free," but instead is found in a complex interaction between embodied perception and the social world. Therefore, despite the embodied nature of mathematics, there are means by which able people can be deterred or excluded from communities and their resources, which would otherwise enable them to participate in mathematics or form identities adequate for participation. However, experiences through which mathematics is directly conveyed as "grounded" in embodied realities can serve as identity construction and affirmation for people as learners of mathematics.

In this work, I describe *conceptual metaphor games*, a form of mathematical learning game which is (1) grounded in the embodied cognitive science of mathematics and (2) suitable to be implemented as a practical and person-centric approach to mathematics pedagogy.

Literature Review

In setting the stage, historical context is presented for the discipline of cognitive science and the theory of embodied cognition, along with a discussion of the application of cognitive science methods to inquiry regarding mathematics. Then, a general summary of the driving pedagogical theory behind the work is offered, before going on to describe *conceptual metaphor games*.

Cognitive Science

Cognitive science is an approach to studying "mind" which is classically defined as the intersection of six disciplines: psychology, neuroscience, anthropology, linguistics, philosophy, and computer science (Von Eckardt, 1993). As the story goes, the history of the discipline of cognitive science is linked to the theory of computation—to the excitement of automata, formal language, and information theory. The so-called "cognitive revolution" coincided with the arrival of electronic digital computers and growing interest in artificial intelligence (Pinker, 2011). While the proposed interdisciplinarity of cognitive science may be exciting, it is worth mentioning that there is an ongoing historical debate over the foundations of the field, in particular with respect to (1) its definition, including its relationship to *computationalism*, i.e. the view that the mind is a computing machine, and

(2) its status as either multidisciplinary or as a truly integrated interdisciplinary science (Gray, 2019; Núñez et al., 2019). To frame the context for this work, we describe some relevant history, including where these foundational issues arose, along with the build-up to the need for an embodied cognitive science.

Cognitivism is often contrasted with behaviorism, notably in the words of psychologist George A. Miller:

> Behaviorism was an exciting adventure for experimental psychology but by the mid-1950s it had become apparent that it could not succeed. As Chomsky remarked, defining psychology as the science of behavior was like defining physics as the science of meter reading. If scientific psychology were to succeed, mentalistic concepts would have to integrate and explain the behavioral data. We were still reluctant to use such terms as 'mentalism' to describe what was needed, so we talked about cognition instead. (Miller, 2003, p. 142)

However, while *mental representation* is generally an object of study in cognitive psychology, cognitive scientists do not swear an oath to shield their eyes from human behavior. That is, cognitive science is not properly defined as mutually exclusive from the behaviorist project. Instead, one succinct summary of the underlying suppositions of cognitive science is that "thinking can best be understood in terms of representational structures in the mind and computational procedures that operate on those structures" (Thagard, 2020), but there are varying ideas of what "representation" and "computation" are. Barbara Von Eckardt's *What is Cognitive Science*? outlined four important components which make up the cognitive science framework: "the domain-specifying assumptions, the basic questions, the substantive assumptions, and the methodological assumptions" (1993, p. 303). In that work, she certainly emphasizes the importance of mental representation and the computationalist view, although from its inception in the mid-20th century, the latter idea has been as popular as it is controversial.

Just what "computation" is and what its definition implies about "mental representation" is not a trivial matter. Borrowing from the work of C. S. Peirce, von Eckhardt describes a common notion of representations, consisting of three parts: *sign*, *object*, and *interpretant* (Von Eckardt, 1993, p. 145-159). For example, a person may form a mental representation of a friend. Roughly speaking, how the representation of the friend shows up in the brain and body of the person is the "sign"; the "object" is the person themselves; and the "interpretant" could involve, for example, the emotions the friend evokes in the person. But this hardly says anything about what signs are or how they come to represent objects.

Since the computational view assumes that the mind is a computing machine, representations are generally understood in terms of "information-bearing structures" that process a kind of "machine code." On a classical view such as Jerry Fodor's *Language of Thought Hypothesis*, the "machine code" processed is a kind of "mentalese," the mind's way of encoding information in a way that resembles language (Pitt, 2020). This idea, which posits thought to be "atomic" in that there are some minimal linguistic units which compose thought, is reasonably fleshed out to the point that it gained significant traction as an idea worth taking seriously (Von Eckardt, 1993, p. 342). However, a closer examination of the structure of nervous systems gives rise to a counter view known as *connectionism*. Connectionists emphasize the "rhizomatic," interconnected structure of neural architectures, which perform computations in a highly parallel way. This means that the overall distribution of processes occurring throughout a brain-body system play into "representation," not just local neurons that do serial abstract computations. So on this reading, instead of discussing mental symbols, the subject matter is *patterns of activation* in neural systems.

Historically, though, the computational metaphor has been important to the study of cognition, regardless of how seriously it ought to be taken. For example, cognitive scientists may choose to "dissect" the mind into mechanical components: perception, memory, attention, judgment, and so on, although taking these faculties to be literally distinct may amount to modern-day phrenology. Today, one has to take care in describing exactly what is meant when describing the mind as a computational machine; it is often held that the mind-computer analogies are at best convenient fictions. While "cognitivists" in the strict sense are still around today, even von Neumann, who spelled out the brain-computer analogy in writing as early as 1958, could not bring himself to accept it as any more than a metaphor (Peters, 2018; Von Neumann et al., 2000). The analogy suffers from internal problems related to both time and memory complexity as well as functional and architectural differences, notably including the fact that the human nervous system operates in a massively parallel way.

From a more general view, this romantic metaphor doesn't look too different from the Cartesian idea that animals are machines—after all, "the spirit of Hebrew clay, the Roman aqueduct, the hydraulics of the humors (and its eventual Cartesian pump), the medieval catapult, Freud's steam engine, Helmholtz' telegraph, and today the holograph, among a host of other new media, have all been compared to the brain and its neural system" (Peters, 2018). In other words, comparisons between mind and computer appear to be no more than particular examples of mappings from tool-related conceptual domains into the conceptual domain of "mind" itself. Peculiarly, the very concept of metaphor has been increasingly recognized as an important facet of human cognition, especially by the philosophy of science, ever since as early as the birth of cognitive science itself (Hesse, 1965; Von Eckardt, 1993, pp. 98-104), and in the context of several newer theories which go beyond classical computationalism under the umbrella of embodiment (Lakoff and Johnson, 1999).

Indeed, it has been increasingly acknowledged that the body plays a central role in mental faculties, an idea referred to as *embodied cognition* (Barrett, 2011; A. Clark, 1999). However, these theories are relatively new and rapidly developing, not necessarily unified, and some more radical or controversial than others. As alluded to, *situated cognition* emphasizes the role of the environment and context in cognitive processing, in part drawing largely on the ideas of J.J. Gibson, especially the notion of *affordances*: environments "afford" certain behaviors from animals. These affordances do not belong properly to the mind or to the environment, but instead are found in the interaction between the two. In addition, perspectives in connectionism and *parallel distributed processing* have demonstrated that cognitive processing is far from serial (Núñez et al., 2019). Margaret Wilson gave an excellent overview of the situation as it was two decades ago in her "Six Views of Embodied Cognition" (2002). Despite its recency, the conceptual framework of embodiment is important to the central idea of this work, and is discussed in more detail in the next section.

Today, as the foundational challenges persist, some question whether there is hope for a truly integrated "cognitive science" as opposed to plural "cognitive sciences." Namely, Rafael E. Núñez, an influential figure in the modern cognitive science community and coauthor of the book *Where Mathematics Comes From* which lays groundwork for this thesis, recently published a paper expressing concern that the integration of the cognitive sciences is a dead dream, and that overall the project of cognitive science has failed (Núñez et al., 2019). The paper argues that psychology and computer science have largely dominated publications in cognitive science, while the other four "hexagonal" disciplines are largely left behind, especially anthropology. On the other hand, that original paper, which employed digital methods to generalize over a vast body of academic literature spanning over more than half a century, has received a noteworthy amount of criticism (Gray, 2019). In fact, some hold the opposite view, i.e. that cognitive science is "thriving on both the research and educational fronts" and that "it shows great promise for the future" (McShane et al., 2019). These debates, in reality, are nearly as old as the field itself; even von Eckardt cited 20th-century examples of concerns that "perhaps there is no such thing as cognitive science, really," and in turn spends the rest of her book arguing that "there is far more implicit agreement among cognitive scientists (of all disciplinary stripes) as to their goals and their basic assumptions than the skeptics would have us believe, and that where genuine disagreement exists—and it certainly does on a number of points—there are rational grounds for adjudicating it" (1993, pp. 1-3).

A thorough untangling of this dizzying braid of theoretical back-and-forths is beyond the extent of the matter at hand. However important these matters may be, the present work operates under the assumption that the practice of cognitive science is at least possible, and makes a genuine attempt towards interdisciplinarity. Additionally, while the importance of the computational metaphor fades under the shadow of embodied processing and situated cognition, these worlds are not necessarily all distinct for some conscientious definition of "computation."

Embodied and Situated Cognition

One way to state the embodiment hypothesis is that sensory and motor processes are tied up with cognitive processes, and can serve at least to some extent in place of abstract disembodied representations such as "mentalese." Embodied cognition can maintain discussion of "representation," but on this reading, representation occurs through sensorimotor activation patterns rather than being separate from them. For example, Wilson gives the example of how neural patterns of activation associated to the visual, auditory, and kinesthetic memory look similar to that observed when the recalled experiences are actually perceived (2002, p. 633). In other words, she argues that many "offline" processes that appear highly abstract, such as mathematical activity, in fact are co-optings of sensorimotor networks, "decoupled" from "real" input. In place of a model of the brain as a central processing hub that commands the body by internally representing every aspect of the world in a "perceive, think, act" fashion, instead embodied cognition is a dynamic coordination between perception and action that runs "just-in-time" (A. Clark, 1999), and while offline processes do occur, they occur through the body. Moreover, in this view of the mind, temporospatial properties of bodies serve as parameters of "processing" whereby living entities perceive and navigate the world.

For example, in a behavior known as "phonotaxis," female crickets orient towards male crickets that have the loudest songs. Performing a sorting algorithm would appear to be beyond the neural capacity of the average field cricket. In fact, no such algorithm is needed. As demonstrated by Barbara Webb and her biomimicry "cricket robots," the "program" is really just "go wherever the sound gets louder." This process occurs through the neurally and vibrationally coupled structure of the cricket's auditory system, along with the males' auditory signals being staggered to the rhythm of the refractory period of the interneurons that, by design, can respond directionally to input (Barrett, 2011, pp. 45-49). This example is a clear case of the importance of an organism's body in a process which would otherwise be considered "cognitive," and, as will be discussed, similar generalities extend to analyses of human behavior. What is more is that, as the picture of cognition expands to include the body, it is not too long before one arrives at the necessity to go even beyond the skin; that is, the mind is influenced by things *external* to the body.

Situated cognition and distributed cognition both hold that an organism's environment plays an influential role in shaping both perception and action. Environments afford certain actions from organisms in particular contexts, notably through the associative structure of neural systems. For example, for many humans, a pen affords writing; it "calls out" to be written with. The theory of distributed cognition emphasizes the blurriness of the lines dividing individual minds and their environments. When a pen is in grip and writing ink on a page, the pen and paper in a sense become a "part of" the mind and body of its user as they work together. In the example of solving a math problem, the brain-body system need not store an accessible representation of the written information, the cognitive burden of which is *offloaded* onto the paper. The strong view that the mind need not be internal and that the environment can be considered part of cognition is known as *extended mind* (A. Clark and Chalmers, 1998).

As mentioned, the perspective of situated cognition is relevant here insofar as it places the study of mathematics in an inherently social context. From the work of cognitive anthropologist Jean Lave and educational theorist Étienne Wegner, a *community of practice* is “a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice,” and it is “an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage” (1991, p. 98). That is, knowledge is necessarily a community practice rather than a static, intangible body of propositions; see also Kuhn’s *Structure of Scientific Revolutions*, where he consistently speaks of “scientific communities” (1970). This draws attention to the fact that, even with its embodied origin, much of knowledge must be obtained through cultural reinforcement, and so social elements can serve as inhibitions in the learning pipeline. In the particular case of mathematical knowledge, the same general facts apply.

Embodied Cognitive Science of Mathematics

At a general level, one might argue that if cognition is embodied, and mathematics is in some sense cognitive, then mathematics is also embodied. To give a more specific example of this genre of discourse, consider “An Extended Mind Perspective on Natural Number Representation,” where Helen De Cruz gives an analysis of numerical representation in humans, in particular the relationship between bodies and counting (2008). Notably, she mentions the importance of body-part counting in many cultures, citing facts ranging from the relationship between the English words “four” and “five” and the Proto-Indo-European words for “finger” and “hand” respectively (note also the relationship between the convention of ten fingers, the decimal system, the English word “digit,” and the Latin word *digitus*), to the more complex body-part counting systems of the New Guinea Highlands in which touching an individual body part corresponds to a certain quantity. She highlights the practical convenience of the metaphorical mapping from the *body schema*, a collection of neural encodings of the body’s spatial configuration located in the left intraparietal region. The closeness of these neurons to the inferior parietal region, which has been repeatedly shown to be a hotspot in the brain associated with mathematical activity, makes for easier communication between the body schema and the conceptual domain of arithmetic.

Representation of Quantity

Twelve years prior, cognitive neuroscientist Dr. Stanilas Dehaene popularized ideas in the cognitive science of mathematics in the 1996 edition of *The Number Sense* (2011).[1] He speaks a fair bit of the aforementioned inferior parietal cortex (pp. 189-190, 228-230), a region in which the presence of lesions can lead to complications with mathematical ability. For example, the book surveys classic theories about the representation of quantity in animal cognition. A well-replicated result, known as early as 1886 (Dehaene, 2011, p. 66), reveals that humans can immediately "subitize" cardinalities one, two, and three, although it generally gets more difficult starting at four (Mandler and Shebo, 1982). From the Latin *subito* meaning "suddenly," subitization is the "instant" recognition that a certain number of objects are present, where "instant" means roughly around half a second (Lakoff and Johnson, 1999, p. 19). There is also extensive evidence that non-human animals, including fish (e.g. *Poecilia reticulata*, also called guppies) and honey bees (*Apis mellifera*) can work well with quantities 1-4 (Agrillo et al., 2014; Dacke and Srinivasan, 2008). Subitization is distinct from explicit counting or estimating; the former is a uniquely human "algorithm" which employs a recursive pattern to assign quantities to objects, allowing one to work with numbers which they cannot otherwise conceptualize. Without counting, precision and speed of recognition fall off drastically after four, as subitization rapidly turns into approximation. The acknowledgment of this distinction led to a search for the cause of the

[1] His work is reminiscent of the witty writing style of the logician Raymond Smullyan, who loved logic puzzles. Speaking of twelves and Raymonds, the book opens with a quote by Raymond Queneau: "any poet, even the most allergic to mathematics, has to count up to twelve in order to compose an alexandrine" (p. 3).

phenomenon.

Dehaene wrote in support of a theory about a mechanism of *mental accumulators*. An accumulator can be imagined as an electrical circuit which "accumulates" intensity as a signal flows across its input channel, and when the signal stops, it outputs whatever the final summed intensity was. It acts an approximate measuring machine for sensorimotor signals. This idea relates to *Weber's law*, which states that there is a relationship between the pre-existing "intensity" of something perceived and how much change needs to occur in what is perceived in order for it to be recognizable (Dehaene, 2011, p. 72). When looking at a page filled with three dots, it is easy to tell it apart from a page with ten dots; when there are 200 dots, one may need to add some dozens of dots before a change is detectable. In mathematical terms, the minimum change ΔS in intensity required for a change in stimulus to be perceptible is a constant percentage k of the intensity S of the stimulus, that is $\Delta S = kS$. By the tradition of the similar but logically independent *Fechner's law*, the mental accumulator does not accumulate as fast as the input signal itself; its output scales as the logarithm of the measured intensity S (Algom, 2021). Given that this relationship has empirical evidence, it suggests the human "mental number line" is not linear but approximately logarithmic (De Cruz, 2008). Yet, the previous paragraph demonstrates that Fechner's law fails at low intensity; perhaps the error in the "mental accumulator" mechanism is highly negligible for lower numbers, although on the other hand, one might argue that they are separate mechanisms altogether (Agrillo et al., 2014).

On a final note with respect to the research of Dehaene, it is worth noting that an effect has been observed in experimental psychology regarding human associations with "large" and "small" versus "right" and "left." Referred to by Dehaene as the *Spatial-Numerical Association of Response Codes*, or SNARC in honor of Lewis Caroll's nonsense poem "The Hunting of the Snark," the finding concerns the fact there are various scenarios where human samples can demonstrate a directional bias in association between magnitudes and spatial information. First, when American subjects were asked to press a key to indicate whether a number is larger or smaller than 65, it is reported that the effect exemplified itself as a faster response with the right hand when the given number was larger than 65 than with the left hand, and vice-versa for when the number was less than 65 (Dehaene, 2011, p. 80-81). Second, while SNARC has been observed in a variety of contexts and replicated over a hundred times, there is also reasonable evidence that this phenomenon is subject to cultural influences; Dehaene says that in a sample of Iranian subjects, those with “less exposure to Western culture” tended to demonstrate an association of larger magnitudes with “left-space” instead (2011, p. 82). This implies that, among a variety of variables, directionality of writing systems may be a factor, but these claims deserve a very careful analysis. Last, as a SNARC effect has been observed for auditory signals, it is possible that it is an amodal phenomenon (Nuerk et al., 2005). While the details require further investigation, the effect indicates an important relationship between spatial frame of reference and conceptions of quantity.

Schema and Metaphor

Both influential and controversial, published at the dawn of the second millennium, a book which elaborately popularized an embodied theory of mathematics is the mentioned *Where Mathematics Comes From* by cognitive scientist Núñez and cognitive linguist Lakoff (2000). The authors argue that the foundations of mathematics cannot be described by mathematics itself, but instead can be reduced to embodied metaphors through a process called *mathematical idea analysis*. Citing authors such as Stanilas Dehaene and Saunders Mac Lane, the book characterizes counting and arithmetic with “Four Grounding Metaphors,” logic and sets with spatial relations, and infinity and the real numbers with the “Basic Metaphor of Infinity.” In a 450-page analysis, the authors paint a polarizing picture of mathematics, supported by evidence from Lakoff’s theory of cognitive metaphor, embodied neuropsychology, the history of mathematics, and of course mathematics itself. This section attempts to shed some light on how their work looks at abstraction through an embodied lens.

Although some have raised the point that one probably ought to be skeptical of taking any given metaphor at face value (Sinclair and Schiralli, 2003), there is a surprisingly convincing contention that metaphor is central to everyday and abstract thought (Lakoff and Johnson, 1999). Consider the ubiquity of phrases which compare commonplace matters to physical properties of the world. When two things are similar, they are "close"; when they are different, they are "far." In the case of emotional "distance," one is "cold," whereas there is a comfortable "warmth" or "heat" to attraction. These relationships may appear to be no more than linguistic happenstance, but convergent evidence suggests that they are not entirely arbitrary. In some cases, the correlation between the two relevant domains is such that the domains become *conflated*; whenever one neural pattern of activation occurs, the other shows up as well, an event referred to as *coactivation* (Lakoff and Núñez, 2000, p. 43). Although real neural systems get more complicated than just the general Hebbian maxim that "neurons that fire together wire together," this fact illustrates neural regions which commonly coactivate become both electrically and vibrationally linked. This is a simple intuition for the causal mechanism behind conceptual metaphor.

Some housekeeping is in order. So far, there have been references of *conceptual domains*, which have not been defined yet somehow treated as though they were actual objects in a Platonic sense; this need not be so. Whatever the ontological commitment, the key assumption is just that conceptual domains are organizations of knowledge, so that it makes sense to discuss things such as a person's "conceptual domain of arithmetic." Moreover, note that while Lakoff and Núñez's book *Where Mathematics Comes From* makes radical claims about the universal applicability of the concepts mentioned here, there has been extensive literature critically analyzing those claims; for example, the book speaks little to the ways particular individuals represent concepts, instead presenting its analysis as the "real" way mathematical concepts are understood (2000; Sinclair and Schiralli, 2003). All this said, arguments for some key takeaways from their analysis are made here. To begin, it is necessary to introduce a new concept, instances of which will present as unified nouns, but again it is assumed that these "objects" are simply useful concepts which have some manifestation in brain-body systems.

Lakoff and Núñez posit that structures referred to as *image schemas* (also spelled *schemata*, as it is from Ancient Greek σχῆμα, "form") play an important role in the cognitive organization of information by integrating multimodal data, providing communication between different faculties (Lakoff and Núñez, 2000, p. 27-49). To give an important example, the *Container* schema is composed of an Interior, an Exterior, and a Boundary[2]. Together, these parts form a mental model such that one can be "highlighted" or *profiled*. So the example of the In schema, say, for a person being in a room, the Interior of the room is profiled, and it serves as the landmark for the person. A fascinating entailment about schemas is that they are important to perception; after all, when something is said to be "in a field," where could the container of the field be but in the imagination? In this vein, cognitive linguist George Lakoff has demonstrated that there is a plethora of data from the study of language informing the concept of schema, including the insight that image schemas are not necessarily universal and may vary across cultures and languages (Lakoff and Johnson, 1999).

The Container schema is a classic example offered by Lakoff and Núñez (2000, p. 31-32) that connects the internal relations of the spatial schemas to the foundations of logic and set theory. A set in mathematics is a collection of objects, and the principal relation for sets is membership; in English, when this relation holds between a containing set X and an element x, one says that "x is in X," and writes $x \in X$. For example, the natural number 2 is in the set of evens. Furthermore, it is generally thought to be a contradiction that something could be both "inside" and "outside" of a container, an idea strongly related to the *principle of excluded middle* in classical logic: Given a statement p, it is either true, or the negation of p is true (although in intuitionist logic, this does not

[2] In the branch of mathematics known as topology, sets are commonly dissected into these three parts.

hold!). So, given that p is “2 is in the set of evens,” in classical logic, it is either the case that 2 truly belongs to the set of evens or that is does not, and in the latter case it may be said to be “outside” the set. In fact, sets are often visualized as Venn diagrams which “embody” their relationship to the Container schema. That depiction highlights the fact that object can be in (or out of) two Container schemas at once; the middle of a Venn diagram is likened to the *intersection* of two sets, the set which contains only the elements which reside in both sets. Before coming to how it is that humans form this connection between the abstract domain of sets and the concrete operations of the Container schema, one other schema must be discussed.

The other relevant schema worth introducing here is the *Source-Path-Goal schema.* This schema is very important to conceptions of motion. Its object is a *trajector* which moves, and the trajector has a *source* at which it starts, a *path* along which it travels, and a *goal* to which it travels. At each point in time, the object has some *position* and *direction*, and as the actual trajectory of the object may or may not be different from its intended path, the goal may or may not be the same as the end location of the trajector. Basic properties internal to *Source-Path-Goal* schemas manifest in a wide variety of mathematical concepts. Lines are thought of as “meeting” at a point, and functions are said to be “increasing” or “decreasing,” as if either of these things are truly moving. For example, bringing an example from abstract algebra, cyclic groups are often visualized as the symmetries of regular two-dimensional shapes such as an equilateral triangle, whose symmetries would correspond to the cyclic group of order three, denoted Z_3. Concretely, rotating such a triangle by 120 degrees makes it “fit” back onto itself, so this is a symmetry of the triangle, and the action of rotation is likened to the generating element of the cyclic group. In this example, the source is the initial orientation of the triangle; the path is the 120-degree rotation it takes; the goal is the new orientation of the triangle obtained after moving. The schema even manifests itself in the fact that there is an explicit object in the cyclic group Z_3 called the *identity element* which refers to the “source,” or the motion which remains at the current source state. So, how is it that spatial understanding of the world gets wrapped up in abstraction?

This is where conceptual metaphor comes in. In general, a conceptual metaphor is whenever information in one conceptual domain is thought of in terms of another domain. An elementary example of such a metaphor important to abstract thought is the metaphor Categories Are Containers. The *source domain* of this metaphor is the Container schema and the *target domain* is the conceptual domain of categories. When one speaks of bats being "in" the category of animals, it happens via the Categories Are Containers metaphor.

So it appears that *How the Embodied Mind Brings Mathematics Into Being*, as the subtitle of their book goes, is prevalently through both (i) metaphorical mappings from the domain of physical phenomena to mathematical domains (*grounding metaphors*), and (ii) higher-order mappings between abstract domains (*linking metaphors*). For the conceptual domain of arithmetic, there are some important grounding metaphors from physical domains, which serve to generalize over a range of phenomena and motivate foundational concepts. In Lakoff and Núñez's original theory there are four main ones, but again, there is no rule that there are only four; all that matters is the general framework they provide. Their tetrad consists of Arithmetic Is Object Collection, Arithmetic Is Object Construction, Numbers Are Physical Segments (the Measuring Stick Metaphor), and Arithmetic Is Motion Along a Path. Since this work treats the final of these four in what follows, I only extensively elaborate on that one, and leave the construction of the other three as "exercises for the reader."

Arithmetic As Motion

In what follows, two ways of conceptualizing the metaphor of arithmetic as motion are presented. The first is in the more or less informal spirit of *Where Mathematics Comes From*, where aspects and properties of the source domain are connected to those of the target domain. The second is a more "formal" characterization of the same metaphor, presented in a similar light to the concept of *homomorphisms* in algebra, or *morphisms* in category theory. Both perspectives may be useful for different purposes.

Source domain *Motion Along A Path*		**Target domain** *Arithmetic*
The action of moving along a path	⟶	Arithmetic operations
A point-location on the path	⟶	The result of an arithmetic operation; a number
The origin, the beginning of the path	⟶	Zero
The unit location, a point-location distinct from the origin	⟶	One
Further from the origin than	⟶	Greater than
Closer to the origin than	⟶	Less than
Moving from a point-location A away from the origin, a distance that is the same as the distance from the origin to a point-location B	⟶	Addition of B to A
Moving toward the origin from A, a distance that is the same as the distance from the origin to B	⟶	Subtraction of B from A

Above is a reproduction of the table representing the Arithmetic Is Motion Along a Path metaphor from *Where Mathematics Comes From* (Lakoff and Núñez, 2000, p. 73). Here, “arithmetic” concerns natural numbers, but the authors mention that the metaphor is readily extended to negative numbers. The authors go on to show that these same ideas can be extended to multiplication and division—in the sense of “skip motions,” i.e. instead of moving one unit at a time, one skips over some each time—as well as fractions, by only moving a portion of the unit motion. In a similar vein to the other mappings mentioned in the book, one can also draw connections for particular properties of the domain, for example, commutativity: in the same way that the order of addition doesn't affect the outcome, the order of linear motions doesn't affect the end result, as "moving three left, and then two right" is just as good as "moving two right, and then three left."

The benefit of drawing out a table like the one above is that it makes the nature of the mapping clear by showing how the different elements of the source domain match up to the target domain. However, its convenience has a price in another sense: it justifiably posits a generic domain of "arithmetic," which is more faithful to the conceptual realm than it is to the modern field of mathematics. For example, in mathematics, notions of "greater than" or "less than" are often considered belonging to the domain of *order theory*, whereas the content of the arithmetic operations mentioned properly belong elsewhere. If one interprets "arithmetic" to be the integers rather than just the natural numbers, the realm of interest is *group theory*, as it is characterized with the theory of abelian (i.e., commutative) groups, specifically dealing with the additive group of integers $(\mathbb{Z}, +)$. (For readers unfamiliar, "the additive group of integers" is a fancy way of referring to the mathematical object obtained by taking all "whole" positive and negative numbers along with zero and equipping this set with the operation of addition.) We could attempt a more minimal mapping, as shown below.

Source domain *Motion Along A Path*		**Target domain** *The Additive Group of Integers* $(\mathbb{Z}, +)$
The actions of moving uniformly along a path	$\longrightarrow$	Elements of the group
The action of “doing nothing”	$\longrightarrow$	Zero (0), the identity of the group
The unit action in a direction along the path	$\longrightarrow$	One (+1), the generator of the group
The unit action in the opposite direction along the path	$\longrightarrow$	Negative one (-1), the inverse of the generator

Note, first, that this view of the metaphor uses a small amount of information to demonstrate how the two domains correspond. In a sense, one could limit to just the last two rows (or perhaps even just the third), but all four are included for clarity. All other correspondences one could draw out between the two domains follow from these. the version of the metaphorical mapping in the table above may possibly be improved, albeit perhaps made less clear, by referencing the conflation between action and state. An element of a group is generally thought of as an action or a transformation. However, it is also a “state”; the number one may be the act of hopping once on a path, or it may be the state obtained after performing such an action. In mathematical terms, the preimage of a group element under the metaphorical mapping is both physical motion and the result of physical motion. Either way, this pseudo-formal mapping can be thought of as a “proof” that every internal fact about the additive group of integers can be rewritten as a fact about linear motion along a path. However, it seems easily argued that even this particular metaphor is still more heuristic than rigorous; for example, one needs to conceptualize an infinitely long path for “every fact about the integers” to directly translate. Nonetheless, this is reminiscent of the concept of *homomorphism* in algebra, where the image of the generators of a group under a homomorphic mapping determines the entirety of the mapping.

A *homomorphism*, informally, is a mapping from one object to another which in some sense respects the structure of the object being mapped. This concept is ubiquitous in grade school mathematics, just not drawn attention to. For an example, consider the set $\mathbb{R}$ of all real numbers equipped with the operation of addition $+$, and let c be any nonzero real number. Let ϕ be the function (a "rule") which sends any real number x to $\frac{x}{c}$; in other words, give ϕ a number x, and it will divide it by c. The map ϕ is a group homomorphism, since for any real numbers x, y, we have

$$\phi(x) + \phi(y) = \phi(x + y)$$

or more explicitly

$$\frac{x}{c} + \frac{y}{c} = \frac{x + y}{c}.$$

In words, this means that "adding x and y and then dividing by c" gives the same result as "first dividing x by c, then dividing y by c, and adding them." This is not necessarily an obvious fact; it is a nice property of the real numbers. As in the case of any homomorphism, where ϕ sends 1 determines the entire mapping; e.g., if ϕ sends 1 to $\frac{1}{2}$, then every real number gets halved by ϕ. What is more is that, since ϕ uniquely sends every real number to every other real number, i.e. it creates a *one-to-one correspondence* from the set of reals $\mathbb{R}$ to itself, this map is called an *isomorphism.* (In fact, because it is an isomorphism on the group $\mathbb{R}$ itself, it is an *automorphism*, a "self"-isomorphism.) When two groups are isomorphic, they are in a sense "the same group" in that their elements have exactly the same relations.

This is all worth summarizing because the similarity between homomorphisms and metaphors is meaningful in this context. For one, the examination of these matters is that of—surprise!—a metaphor itself, namely the metaphor Metaphors Are Group Homomorphisms. In the case of the Arithmetic Is Motion Along A Path metaphor, the relationship does not hold up well; no one has ever proved the existence of an infinitely long path in the real world, but the integers form an infinite set. However, when one restricts to just a few lily pads hopping in a circle, one encounters a different, much more precise metaphor: Modular Arithmetic Is Motion Along A Cyclic Path. The term *modular arithmetic* refers to how counting works on a clock: starting at one, going up, and resetting back to one after twelve. In this case, the formal target domain is a *finite cyclic group*, whereas before, in the case of the integers, it was an infinite cyclic group. This idea will be made concrete in the elaboration of the methods, but now the paper turns to general pedagogical matters.

Pedagogy and Mathematical Ability

What puts limits on human aptitude for mathematics? There are some popular answers to this question which are found lacking. A dominant stance is that the abstract nature of mathematics is what makes it so difficult that it is suitable for only those blessed with the elite intelligence required to understand it. This is a kind of "fixed-trait" view that attempts to takes into account biological variance in mathematical ability. Analysis of responses to an attitudinal questionnaire presented to 284 primary students in Greater London revealed that many students attributed mathematical ability to fixed characteristics such as being "cleverer" or having a bigger brain (Marks, 2015). Their answers demonstrate a reconciliation of identity with self-perceptions in relation to mathematical ability, the common belief being that mathematical ability or the lack thereof is simply a matter of "who people are."

Indeed, the culture of mathematics itself appears to be one of sensationalism, glorifying the "geniuses" with remarkable insight. Besides the some hundred of concepts named after the mathematician Gauss, equally common is talk of the "divinatory" analysis of Srinivasa Ramanujan, who lived during the time of British Raj and is praised for his plentiful contributions to mathematics, despite coming from a background of little formal mathematical education. A biography of his life details a legend where a friend of Ramanujan, K. S. Srinivasan, went to see him. "Ramanju," his friend said, "they call you genius." "Hardly a genius," said Ramanujan, "Look at my elbow, it will tell you the story... my elbow is making a genius of me" (Kanigel, 1992, p. 92). Ramanujan's elbow, blackened from using it to erase his slate which helped him work on mathematics day and night, is a symbol of his persistent spirit which led him to "befriend every natural number." The moral of the story is that, although undoubtedly both innate and nurtured factors play into just about any complex trait imaginable, even Ramanujan humbly acknowledged the important role of practice. While historical icons like him are certainly noteworthy, it leaves to dispel the idea that mathematics is only "for" people like them.

For one, as there are metaphors which develop between abstract domains and even among metaphors themselves, the content of human mathematics readily becomes so abstract that the extent to which it preserves the genuine structure of the physical world dwindles. Although embodied intuitions underlying mathematical ideas are often simply disregarded, abstraction does play an indispensable role in both the beauty and notoriety of mathematics. While that fact is part of what makes mathematics challenging for many people, and perhaps shockingly easy for some few select prodigies, it is not the full story. Not all of mathematics is so abstract as to be divinely alien, and in the event that it is, it involves the exploration of "possible worlds" through and in relation to knowledge which is fundamentally embodied—for example, as posed, there is a good case that logic itself has its basis in spatial intuitions about the world. On an empirical note, there is evidence from functional Magnetic Resonance Imaging (fMRI) that in both hemispheres of the human brain, the "prefrontal, parietal, and inferior temporal regions" which consistently activate during numerical processing tasks, is recruited just the same when professional mathematicians do higher mathematics (Amalric and Dehaene, 2016). Combined with the finding from the same research that language-related faculties do not seem to play an important role in any of these mathematical tasks, their research implies a relationship between prelinguistic embodied numerical intuitions and facility with advanced mathematics. So aside from a pedagogical sentiment that a strong mathematical foundation is key, this supports the claim that even the most abstract or "disembodied" mathematics seems to be inevitably connected to bodily processing. Can one therefore claim that only those with special bodily characteristics can understand mathematics?

A strongly affirmative answer to that question would be disingenuous. Surely, there are exceptional cases where mathematically competent people have experienced unfortunate conditions which demonstrably interfered with their ability (Dehaene and Cohen, 1997; Dehaene and Cohen, 1991). For example, "Mr. M," as he is dubbed by Dehaene, suffered a lesion to the inferior parietal region, which created a double dissociation: He was able to perform simple additions, but he could not perform as well on tasks related to subtraction or the ordering of numbers. Furthermore, there are some horrible cases where mathematical activity induces seizures in otherwise healthy individuals[3] (Dehaene, 2011, p. 191-192). On the other hand, in general, there are cases where variation in bodies corresponds to differences in sensory perception which are part of the human experience, but may lead to media being inaccessible (Zambo, 2008). All of this is relevant because they are clear cases of when there are literal bodily constraints on mathematical capability; biological heterogeneity should not be ignored. However, I argue that the summed explanatory power of these reasons is not adequate to account for the disproportionate fraction of able people who sincerely believe they are not capable of doing mathematics.

If one is inclined to accept the theory of embodied cognition as it applies to mathematics, then one should also accept there is a sense in which the theory highlights the blatant discrepancy between commonly held beliefs about mathematics and what ought to be true about mathematics. To spell this out: If mathematics has an embodied origin, then it is about perceptual generalities or invariants across embodied entities, each with a unique and distinct body, yet these bodies are similar enough that a vast number of people can identify, agree on, discuss, and explore the nature of said invariants. That is a

[3] I know personally someone who was very interested in mathematics, but could not continue as a result of such an affliction. It is a deeply upsetting matter.

remarkable story which mathematics tells: Even though humans are all so diverse (as must be recognized more and more), there are elements of individual human experiences that echo across sensory modalities and serve to unify those experiences.

Therefore, an important consequence of this is that mathematical ability is not a special gift which is reserved for Ramanujan, Gauss, or Euler, however gifted they may be. It is not reserved for the clever prodigy or the white male elite, however historically advantaged they may be (Loewen and Others, 1988). Instead, mathematics is something which can be engaged with by people all around the world, of diverse bodies and backgrounds. While it is fruitless to search for a single thing which brings humans together into a monolithic category, mathematical ability comes close to being something truly human. That which mathematics is about came before human experience; it exemplifies itself in non-human animals and the natural world. So what prevents human bodies from engaging in mathematics? Sure, there is diversity in interests; not everyone has to be a “math person.” Regardless, for this to be a meaningful explanation of the explananda, there must be some analysis of why people become uninterested in mathematics. It is not just pedagogical challenges but also inequity in academic opportunity and stigma regarding the subject which prevent capable people from believing in their mathematical ability.

These inequities run deep. There are clear examples of experiences which can serve as identity fragmentation for young learners of mathematics, especially for vulnerable, marginalized populations. To give one example, in a video which proves difficult to watch, a first-grade teacher is heard harshly reprimanding a young Black girl for incorrectly answering a mathematical question, asserting that “nothing infuriates [her, the teacher] more than when you [the student] don’t do what’s on your paper,” and afterward asking the class to show her the correct answer (Times, 2016). Nicole Joseph analyzes this incident in her paper on “Black Femininist Mathematics Pedagogies” where she says that such situations are “unfortunately... not an anomaly” (2021, p. 85-86). Her work sheds light on the ways that Black girls are taught that they do not belong in mathematics and are not provided appropriate care or resources that would foster their mathematical identities. She offers that a solution which promotes “robust math identities” through “ambitious math instruction,” “critical consciousness and reclamation,” and “academic and social integration.” Though contending that it is more than just lighthearted, rigorous instruction that will do justice to underserved communities, she invites readers to “teach Black girls more mathematics, not less.” In America, the project of bringing mathematics and science literacy to economically disadvantaged students and students of color has been around for at least as long as the Civil Rights Movement; see, for example, the Algebra Project, founded by Bob Moses (Moses and Cobb, 2002). Since mathematical familiarity can confer economic advantage if it leads to opportunity in high-paying jobs in applied fields, keeping people from enjoying the benefits of mathematics altogether is disempowering and unjust.

Though there are material reasons to bring mathematics to more people, it may not be obvious that mathematics pedagogy is worth pursuing in and of itself. As an answer to this question, Francis Su’s recent book entitled *Mathematics for Human Flourishing* argues that “deeply human themes” motivate people to do mathematics: exploration, play, and beauty, to name a few (Azusa Pacific University - Lectures, 2020; Su, 2017). In a lecture on his work, he dares listeners to question who they think of when they ask themselves, “Who does mathematics?”. His question serves as an invitation to expand commonplace notions of who gets to participate in mathematical communities of practice.

Mathematical Learning Games

In conclusion of the literary review section, some words on pedagogical mathematics games are necessary before moving on to discussing a game-based solution. A systematic review of literature on learning games for grade and undergraduate school students reported a significant difference in metrics of student learning among game and non-game conditions, with the data favoring digital learning games as enhancing learning (D. B. Clark et al., 2016). Just as this study emphasizes that specific elements of the game implementation, such as using augmented reality (AR) or requiring repeated play, can affect learning outcomes, more generally, a very recent study oriented specifically towards mathematical games investigated the role of design quality on learning outcomes (Bullock et al., 2021). The results of their analysis led to the conclusion that the quality of design features of the game were closely related to the player's awareness of the features, and together these variables influence how a student benefits from the game. Therefore, while learning games are a promising avenue of research, it is important that their details be examined with great care.

Mathematical learning games have been around for years. Possibly the most famous example includes *CoolmathGames*, with the Coolmath network having been founded in 1997 and still around today ("Cool Math Games," 2022). A much more recently popular example is *Prodigy*, which is notably similar to *Pokémon*, except the "dueling" mechanism typically involves some sort of mathematical question ("Prodigy Math Game," 2022). While the game is an excellent means of engaging students with mathematics in a stimulating context, the main content of the game is reasonably separate from the mathematical elements, which arguably serves as a limitation of the game. In an ideal situation, a mathematical learning game would integrate mathematical content into its foundational design while also retaining pleasantness. Making noteworthy strides towards this goal is *ST Math*, whose patented method has been empirically shown to significantly increase mathematical achievement, up to threefold. It is hard to imagine that MIND Research Institute, the creator of ST Math, is as a whole completely unfamiliar with the embodied cognition theory presented here; they claim to leverage "the brain's innate spatial-temporal reasoning ability," and even have a blog post about schemas (Buschkuehl, n.d.; Institute, n.d.).

This concludes the literary review section. Given the current theories about the nature of mathematics and the psychological and anthropological evidence surveying the state of pedagogy, in the next section I demonstrate how to implement this information into an educational game.

Method

The methodology of this work is presented threefold. The first part contains a description of the philosophy behind conceptual metaphor games. The second section surveys a general picture of practical strategies for implementing the games in a variety of contexts. Third, I describe implementation details for a demo level of a virtual game.

Philosophy

The first step to creating a conceptual metaphor game is to identify a metaphor and explore it in concrete terms. In the literature review section entitled “Schema and Metaphor,” the core motivation of the work was written using academic language from the theory of embodiment in cognitive science. However, the embodied nature of these ideas lends them to be illustrated in absence of the underlying theory.

Consider the Frog

Consider the frog who hops on lily pads, laid out in a long line in either direction. As one observes the frog, there are some facts about the frog’s motion which, for many, are nearly impossible to ignore. These are commonly held intuitions which may be considered self-evident, as they are recurring motifs that show up repeatedly in situated human worldviews, echoing through each of the senses, and deeply ingrained in animal biology. To name one, when imagining this situation, it is implicitly assumed that the frog started somewhere. Regardless of where she was before, there’s a sense that there is some *source* of her action, and whenever she acts, she takes some *path* to reach a *goal* (see the description of the Source-Path-Goal schema in the literature review). Another fact is that it seems the frog could hop at any point, at least until she reaches the last lily pad on either side. Alternatively, she could always choose to stay put and rest. A third is that in general there is a sense of direction; the frog could hop left or hop right. These motions are “opposites” of each other in the following sense: Supposing the lily pads are spaced out in a way where each hop only allows her to move a single lily at a time, hopping left after hopping right or vice versa takes her back to where she started. In other words, the combination of hopping left and hopping right is “the same as” as staying put, meaning they have the same outcome.

There are many similar facts. For example, what happens when she hops “left, left, left” and then “right, right”?—it follows from the previous fact that this “amounts to” once left. There are more complicated questions to ask as well; e.g., does the order of her movement matter? Is “hop left, hop left, hop right” the same or different from “hop right, hop left, hop left”? One could draw out some examples to test and, with some carefully spent time, would consistently find that the order does not affect where the frog lands. This may not be an immediately evident fact, but instead one that is discovered to be true of the frog’s motion by empirical observation.

The story expands further: Imagine the frog skips a lily each time they hop. Now this frog is hopping two lilies at a time, and there are some they will never land on until they decide to take a different hop. Or perhaps they skip two lilies, or five. Now the operation at play is multiplication. Suppose a frog only ever skips two lilies with its hops, no more, no less; which lilies will they never be able to reach? Which lilies can they reach that another frog may not be able to?

From the last example, the assumptions made in order to conceive of a situation like this get somewhat ridiculous, but they are certainly imaginable and within reason. For hops that skip one or two lily pads, these stories are physically realizable. Yet, it can be entertaining to think about frogs hopping physically unrealistic distances; intuitions about how things go at small distances extend in a consistent manner to dreams of, say, a frog hopping over a hundred lilies at a time. Since much of mathematics is generalization from “ideal” facts taken to be evident, thinking about such things is in fact natural and can spark joy, taking the form of playful exploratory learning.

Therefore, the general idea is to gamify these sorts of explorations. Wherever a mathematical metaphor reveals itself with clear self-evidence, either in nature or otherwise, there is an opportunity to explore and learn about mathematics "from the source." Take a clear metaphorical mapping from the real world onto mathematics, and make a game out of it. It is possible that similar things can be done with linking metaphors as well, but to start out with, this work restricts to grounding metaphors.

Content

The following section serves as a practical pedagogical guide to implementing conceptual metaphor games for the Arithmetic As Motion Along A Path metaphor, where the source domain of the metaphor involves frogs hopping on lily pads. The study of this particular class of conceptual metaphor games is what I endearingly dub *frog theory.* I first discuss frog theory and then its practical application to education.

To be absolutely clear about terminology, the following definitions are crucial.

- A *set*, roughly speaking, is a collection of objects.
- A *function* is a rule which maps each element in a set to one element in another set.
- The set of *natural numbers* refers to the set $\{0, 1, 2, \ldots\}$ of whole numbers (including zero, although in some cases 0 may be excluded).
- The set of *integers* refers to the natural numbers along with all their negatives: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- The set of *rational numbers* refers to all fractions; i.e., it refers to all numbers of the form $\frac{p}{q}$ where p and q are integers, with q not zero.

Concepts

In everything that follows, imagine a very long line of lily pads upon which frogs can hop. Let the 0-Lily *always* be the lily pad the relevant frog starts on, unless specified otherwise. For any positive natural number n, a frog which skips $n-1$ lilies with each hop is referred to as the *n-Frog*[4], with the extra definition that the 0-Frog hops in place. Then 1-Frog hops one lily at a time, skipping none; the 2-Frog skips a lily with each hop; the 3-Frog skips two lilies each hop; and so on. When there is a frog but how far they hop is not specified, it suffices to just say n-Frog. Let every other lily be "numbered" in the natural way; e.g., the -3-Lily is reached via the 3-Frog hopping once left.

In general, jumping with the frog is an action that has an inverse (i.e., going left vs. going right). This notion of "directionality" is both important in mathematics and well-studied in cognitive science of mathematics, as demonstrated by, for example, the literature on the SNARC effect. For these reasons, notions of *function* and *inverse* are here throughout.

Numbers. Naturally, this metaphor lends itself to the discussion of numbers themselves.

Addition and Subtraction. The accumulation of jumps by a frog is addition. The 1-Frog hopping thrice is the same as the 3-Frog hopping once. The 1-Frog hops three times, and then four times to make seven times. Or the 3-Frog hops once, and then the 4-Frog hops from that place. It can be discovered that the order of addition does not matter, i.e. hops are *commutative*. Below is an example which explores *partitions* of numbers, and it is phrased in two different ways, appropriate for different learners.

- Say some frog starts on some lily pad, then hops right, and then hops right again. If the frog landed on the seventh lily to the right, how far could they have hopped each time?

- Some n-Frog hops from the 0-Lily, and then from the n-Lily, some m-Frog hops. If the m-Frog lands on the 7-Lily, what could m and n be?

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[4] Readers familiar with group theory can recognize the n-Frog to be the generating set $\{n, -n\}$ of the subgroup $\langle n \rangle = \{\ldots, -2n, -n, 0, n, 2n, \ldots\}$.

Here numbers are “being made”: in these examples, seven is being made from different numbers. Alternatively, we can ask what numbers one can “make” as in: Say one sees 2-Frog and 3-Frog. The 2-Frog rides on 3-Frog’s back when 3-Frog jumps, and 3-Frog rides on 2-Frog’s back when 2-Frog jumps. Where can they go together? The solution is all lilies! This corresponds to the fact that every integer can be written as $2x + 3y$ for some integers x, y.

Subtraction comes up in the undoing of addition, and so also in doing algebra with addition. When the 1-Frog hops 4 times, and then 3 times back, it is subtraction which directly gives that they land on the 1-Lily. On the other hand, it is subtraction that tells us that the 3-Lily is a 4-Frog away from the 7-Lily.

Multiplication and Division. Multiplication appears as hopping multiple lilies at once. It comes up in general as counting the number of times one hops. If the 3-Frog hops 4 times to the right, she reaches the 12-Lily (and again, it is the same as the 4-Frog hopping three times).

Again we have the concepts of “making” and “being made”:

- *Multiples*: Say any frog hops alone. Where can they land? For example, the 3-Frog could land on the 3-Lily, the 6-Lily, the 9-Lily, etc.
  - *Meet*: Least common multiple. If the 2-Frog and the 7-Frog hop from zero, where could they meet?
  - *Join*: Greatest common multiple. Does not exist for natural numbers; e.g., 2- and 7-Frog meet every 14 lilies.
- *Factors*: Say a frog hops by herself and lands on some lily. What frog could she have been? (If she landed on the 8-Lily, she could be the 1-Frog, the 2-Frog, the 4-Frog, the 8-Frog.)
  - *Meet*: Least common factor. Always 1 (for natural numbers); the 1-Frog can hop to any lily.

– *Join*: Greatest common factor. At some point a frog is hopping, and she first lands on the 8-Lily, and then the 12-Lily. What's the largest hop the frog could have made?

Every two frogs can eventually meet, provided the faster frog waits around long enough, and the 1-Frog will always land anywhere (as slow as they may be). But some frogs cannot meet faster than their product, e.g., 3-Frog and 7-Frog meet when 3-Frog goes 7 lilies and when 7-Frog goes 3 lilies. Naturally, 3-Frog goes slower than 7-Frog, so 7-Frog must stop and wait on the 21-Lily. Additionally, assuming all frogs start at the 0-Lily, some lilies cannot be reached by any frog along other than the 1-Frog and the lily associated to that frog: e.g., the 13-Lily is reachable by the 1-Frog and the 13-Frog alone. No other frogs can ever get there without help. Therefore a lily is *prime* when it is only reachable by the 1-Frog and exactly one other frog (again, with frogs starting from the 0-Lily, without help).

Modular Arithmetic (Finite Cyclic Groups). Limit the number of lilies to a finite amount, arrange them in some sort of loop, and the same form of metaphor immediately applies to modular arithmetic. For example, if the 3-Frog hops around twice on five lily pads, she'll land on the 1-Lily (i.e., somewhere 1-Frog lands after one hop). In terms of modular arithmetic, this is related to the fact that 6 divided by 5 leaves a remainder of 1, also written $6 \equiv 1 \pmod 5$.

With a prime number of lily pads, one is in the world of *cyclic groups of prime order*. Say there are p lily pads for some prime number p. The 0-Frog stays in place, the 1-Frog goes around all the lily pads in p hops, and the p-Frog hops and lands right back where it started in one hop. Every other frog does an advancing loop of p hops around the entire circle before it lands back where it started. For example, take $p = 7$; seven lily pads. The 1-Frog goes around in seven hops, the 7-Frog goes in one hop. On her first circle around, the 2-Frog skips over the 7-Lily, which is the same as the 0-Lily, and lands on the 1-Lily. That is her fourth hop, and she takes three more hops to land back on the 0-Lily, making a total of $p = 7$ hops. In this arrangement of seven lily pads, there is the same kind of behavior for $n = 2$, 3, 4, 5, or 6. On a group-theoretic level, this corresponds to the fact that cyclic groups of prime order are *simple*, meaning they have no nontrivial proper subgroups.

The following is a way to compute divisibility of a number x by a nonzero number y, i.e., decide whether there is a nonzero remainder when dividing x by y, and if there is, determine what it will be.

1. Choose there to be x lilies in a circle and use the y-Frog.

2. Start the y-Frog on some lily and let them hop as many times as possible without hopping over the starting lily.

3. If the y-Frog has reached the starting lily, then x is divisible by y; otherwise, the remainder is how many lilies are left to get to the starting lily from where the y-Frog ended.

To be more precise about step 3, the remainder r is the number of hops that 1-Frog would need to go from where the y-Frog ended to get to the starting lily. Formally, if the y-Frog is on the z-Lily, where $z = y - r$ and $0 < r < x$, then r is the remainder. For example, take the division of 11 by 5. By step 1, there are eleven lilies. By step 2, the 5-Frog hops twice and lands on the 10-Lily. On step 3, one concludes 11 is not divisible by 5, with a remainder of one.

Mixed Algebra. Some frog has a 3-Frog on her back. She hops three times. Then, the 3-Frog gets off her back, takes a hop, and lands on the 15-Lily. Who was the first frog? The answer is the same as the solution for x in the equation $3x + 3 = 15$, so she must have been the 4-Frog.

Two-Dimensional Frogs (Cartesian Products). Taking the *product* of the set of integers with itself extends the metaphor to the two-dimensional Cartesian plane. Now there is not just left and right, but also up and down. Here, there is a geometric notion of *right angles*, or in general of *orthogonality*: moving up or down does not affect how far the frog has hopped left or right, and vice versa. The products can technically can be extended to arbitrarily many dimensions, but naturally, the further the situation strays from a frog hopping on lily pads in a pond, the less concrete the metaphor becomes.

The decision to allow frogs to hop diagonally can involve the idea that in a square grid, diagonals are longer than straight paths. In the real world, if there's a 1-Frog, they are probably not literally always hopping some fixed unit distance each time, but instead they are called the 1-Frog because they hop one lily at a time. In a grid of lilies, though, one could imagine that the 1-Frog's hop is not far enough to go diagonally across lilies. This could be a way to open up a whole world of analytic geometry, including talk of a $\sqrt{2}$-Frog who hops along grid diagonals, but this is far beyond the metaphor; even starting to talk about fractions gets weird here, as described next.

Rational numbers. There is a possible extension of these ideas to fractions, involving "half-jumps," "third-jumps" and such. However, the fact that this extension does not readily lend itself to interpretation within the context of the metaphor so far indicates that one is stepping out of the realm of integers and into the *field of fractions* of the integers, known as the rational numbers. This may be hard to execute properly outside of a video game context, but there are some benefits to the perspective. For one, if there exists a $\frac{1}{2}$-frog, say, then the benefit is that it emphasizes that fractions are numbers which can be repeatedly counted, like any other. Pronouncing "$\frac{1}{3}$-Frog" as "third-Frog" emphasizes this, since its hops "add up" exactly the same as the 1-Frog, except on appends "third(s)" to the end of where it lands. For example, the third-Frog hopping five times yields five-thirds... of a lily?[5]

[5] *Addendum*: There is in fact a nicer way to deal with rationals that I overlooked in the original thesis, as pointed out by my committee member Dr. Cornel. It is still a different metaphor, but it works nicely: Instead of treating a single frog hop as one unit, instead an entire pond could represent "one," and, say, if there are three lilies in the pond, then one has the third-Frog.

The Collatz Conjecture. This is a bonus topic for the natural numbers: the $3n+1$ problem, also known as the Collatz conjecture. The conjecture is a long-standing unsolved problem in mathematics which predicts that a certain simple process in the natural numbers always stops in a finite number of operations. The player is given a particularly magical frog, and is allowed to move the frog to any lily to the right (positive) of where they started, and when they are ready, the procedure begins:

- If the current lily is even-numbered, the frog hops back half of the way to start; i.e., if she was on the n-Lily, she becomes the $\frac{n}{2}$-Frog and hops once left to arrive at the $\frac{n}{2}$-Lily.
- Instead, if she is on an odd lily, she hops to the $3n+1$-Lily; i.e., she becomes the n-Frog, hops *two* times to the right (as she's already on the n-Lily, so hopping three times would be $4n+1$), and then hops once right as the 1-Frog.

This process repeats, sometimes over and over again, until a certain loop is encountered: If the frog ever reaches the 1-Lily, then her hops will go 1-Lily $\mapsto$ 4-Lily $\mapsto$ 2-Lily $\mapsto$ 1-Lily, and she's back where she started. Therefore, one can say that the process "ends" when she reaches the 1-Lily, because she will just loop from there. The $3n+1$ conjecture states that this process will always have the frog end up back at the 1-Lily, regardless of the initial choice of lily. Since no proof has been offered of this fact, nor a single counterexample which would disprove it, the problem remains open.

Logic and Computation. Since the target domain of the metaphor is arithmetic, it makes sense to discuss numbers, but it equally makes sense to focus on concepts in logic and the theory of computation without stretching the idea, as I show here.

Frog Automata. Many of the proceeding exercises can be implemented in terms of "frog automata." The basic idea is as follows. For this first example, fix a direction for frogs to hop, either right or left. Place frogs on lily pads, and have some frog be the starting frog on an initial lily pad. If the start frog lands on a lily that has another frog sitting on it, the start frog “startles” the other frog, causing them to hop. If that frog lands on another frog’s lily, the same thing happens, and so on, until a frog reaches a lily that doesn’t have a frog.

Figure 1

A simple frog automaton to catch a fly with 2-Frogs.

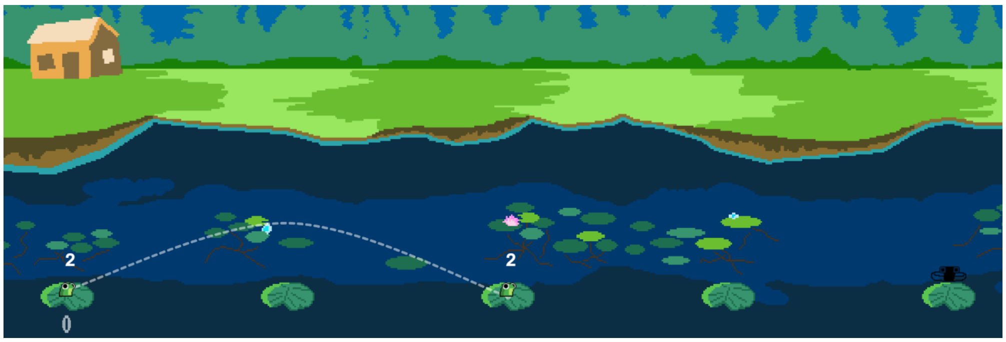


Figure 2 shows an abstract version of a puzzle involving frog automata. The leftmost dot represents the starting lily with a 3-Frog on it. The red dot represents a bug that the frogs want to catch. Assuming frogs only move right, how can one place frogs on the lily pads so that the last frog catches the bug? There are precisely sixteen different ways, each of which is drawn in the figure. One way to interpret the result is that these sixteen ways have something to do with the partitions of the number five, but properly speaking, there are only seven distinct such partitions: 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. A more faithful interpretation of the solution is that it corresponds to the number of ways to select zero or more lilies from a group of four lilies (i.e., the ones between where the 3-Frog lands after their hop and where the bug is), since what one is “selecting” is which lilies to land on. Since, in general, there are 2^k subsets of k items, and in this case $k = 4$, there end up being $2^4 = 16$ solutions. The arrangement of these solutions in the figure below demonstrates the connection to powers of two in that the pattern of grey and green lilies is such that they look like binary strings, starting from the binary number 0000 in the second row from the top and ending with 1111 in the final row.

A brief note is that *nondeterministic frog automata* can be specified by allowing multiple directions without specifying which direction ought to be traveled, or by placing multiple frogs on a single lily.

Frog Puzzles. It is possible to explore different sorts of puzzles. Listing out all the different extensions is not possible, but an example is shown in Figure 3. Suppose this frog is a $\{3, 5\}$-Frog, i.e., it can imitiate the 3-Frog or the 5-Frog, but no one else. This frog wants the fly but she doesn't want to kerplunk into the water! To catch the bug, imitating the 3-Frog alone won't suffice; she'll fall into the hole right before the fly. The 5-Frog overshoots and falls in the water. The solution, instead, is to go right with the 5-Frog, and then left with the 3-Frog, and finally right again with the 5-Frog to catch the fly! This can be presented by allowing the player to "switch" frogs in this fashion, but it also works to set this up as a frog automaton puzzle.

Limits of the Metaphor

Before turning to practical matters, a last bit of theory is reiterated concerning how far the metaphor Arithmetic Is Frog Hops Upon Lily Pads goes. Let a statement p be *internal* to the integers if its domain of discourse is the set $\mathbb{Z}$ of integers, i.e., it only refers to integers and relations over on the integers. Then the theory presented in the previous sections implies that a proposition p internal to the integers can be expressed as a fact about frogs hopping on lily pads provided that p quantifies over (i.e., applies to) only finite sets; unless one grants there could be an infinite number of lily pads. In the case that p quantifies over an infinite subset of $\mathbb{Z}$, the quantifiers of p must be restricted to some finite set. In practice, this does not seem so limiting, and so it turns out that there is a way to discuss just about any fact about the integers by "inverting" the conceptual metaphor and "returning" the fact to a physical source domain.

Practice

In order for the methods presented here to be useful, there must be some guidance on how they are to be used. While it is unscientific to state with certainty what works best without some sort of empirical evidence[6], a working methodology is essential. The content outlined in the previous section is intended to set examples of how to think about conceptual metaphor games in general and how the content gets turned into pedagogical material. After the content is acquired, it must be presented either in the real world or virtually.

Once a feasible conceptual metaphor is identified, one way to go about working with students directly is a threefold procedure. One, the question poser should consider their audience to inform a decision as to what kind of content will be relevant. What should be known about the identities of the students, both as individuals and as learners of mathematics? Second, the presenter should spend some time exploring the content themselves by playing with ideas, forming appropriate ways to ask questions that get at key concepts. Finally, the presenter implements the content into a format which allows the students to engage with it, typically in the form of verbal, visual, tactile, or other sorts of problems to be solved. The particular choice of medium depends on the audience, but one method is a written problem set which the students discuss among themselves or with the question poser.

When exploring frog theory with many students, the proposed method to communicate the rules of the games and ensure understanding at every step thereafter is to work with people one-on-one. Posing problems as lighthearted questions and allowing for a degree of dynamic feedback can help facilitate learning as conversation and exploration. Furthermore, by working with individuals or small groups rather than large classrooms, the presenter can actively adjust the exposition content to be more appropriate and appealing to the audience.

[6] I have firsthand experience working with elementary school students on these kinds of problems. I claim to have had positive results, but the justification of such a claim is beyond the scope of this work.

Frog World

Frog World is a virtual game concept which is based on the ideas presented here. The game involves two main modes: exploration mode and puzzle mode. In exploration mode, the player is encouraged to wander around an "infinite" line or grid of lily pads, where they might find bugs that cure their hunger. On certain occasions, or when the frog gets hungry, the player needs to go on quests which send them into puzzle mode. The puzzles are presented as either click-and-drag frog automata or "free" puzzles where the player is invited to move around their frog directly. During puzzles, the player is encouraged to try different solutions and is shown the outcome of their guesses, e.g., by "running" the frog automaton to see what happens. Upon puzzle success, players are rewarded with experience points and various collectibles, such as new attire for their frogs or frogs themselves.

The game is intended to serve as a kind of *epistemic play* where students learn by discovering the properties of the game world (Hutt et al., 1980). In an ideal version of *Frog World*, the concept could be sufficiently general so as to apply to a wider variety of age groups, varying quest difficulty based on the learner. As demonstrated in the "Content" section, concepts lend themselves to different levels of analysis in the range from concrete to highly abstract.

The embodied design aspects show up in two main ways. First, the conceptual metaphor premise of the game makes for a highly visual-tactile environment which, as a result, relies very little on specific language. Therefore, the game is accessible across language barriers and presents a "friendly" version of a rigorous mathematical metaphor with an anthropomorphic frog. Second, the controls of the game are intended to be gestural in nature. Relevant to the SNARC effect, during exploration mode, the player must press on the right of the screen for the frog to go right, and clicks on the left of the screen to go left. This supports the association of "right" with "greater quantity" and "left" with "lesser quantity"; if desired, the convention can be swapped for people who predominantly write right-to-left. Additionally, in order to hop n times, the player must click n times, thus physically realizing the cardinal aspect of the number n.

Implementation Details. The demo level of *Frog World* was developed as a browser-based game using HTML5, CSS, JavaScript, and Node.js, intended to be accessible to anyone with a device which has Internet access, so that no download or installation is necessary. The game framework used was *p5.js*, “an interpretation of Processing for today’s web” which focuses on accessibility (“p5.js,” 2022). The target audience for the demo implementation is grade school students, especially late elementary (3rd-5th grade). See the appendix for more details.

The features of the demo match the main premise of the game, but they are limited in scope. The demo contains both the exploratory mode and the “frog automaton builder” for puzzles. Three main maps were implemented: “linear”, “linear 2D,” and “cyclic.” The linear mode is the default, consisting of an infinitely long line of lilies (the infinite cyclic group). The second mode implements an infinite two-dimensional grid of lilies, as in the Cartesian product of the integers with itself (product of infinite cyclic groups). Third, the cyclic mode generates lilies in a circle for modular arithmetic (finite cyclic groups). Finally, besides the implementation of puzzles which were described in the previous section, the game also implements a simulatory visualization of cycles from the $3n + 1$ problem.

Discussion

In this work, I have exemplified a method for making pedagogical puzzle games for inspiring interest in and learning mathematics. Besides expounding the theoretical grounding in the embodied and situated cognitive science of mathematics, the last section also provides practical materials which can serve towards the goal of empirically evaluating the method. Importantly, the research conducted suggested that conceptual metaphor can be the motivation for learning games which appear to extend to a very wide variety of mathematical content. Because the games are ripe for low-stress, dialogue-based learning, they can serve as a means to building positive experiences with mathematics and hence the fostering of mathematical identities.

In elaboration of the methods, I hoped to show how “frog theory” is not limited to one level of mathematics, but also explains ideas in abstract mathematics such as group theory. Lakoff and Núñez go much further in their book at explaining how the modern mathematical canon, even showing that the famous Euler identity $e^{i\pi} + 1 = 0$ can be reduced to embodied metaphors. I do not comment extensively here on their analysis of “higher” mathematics, but their general framing is consistent with the neurobiological evidence touched on, i.e., brain networks associated to arithmetic behaviors are recruited also in doing abstract mathematics (Amalric and Dehaene, 2016). It is also common to discuss mathematical content as scaffolding or building upon itself (e.g., Wriston, 2015). All of this suggests the importance of individuals’ comfort with their mental models of mathematics fundamentals.

Given that the ideas here are from the theory of embodied cognition, one might expect that their implementation is embodied in a strong sense. The term “embodiment” here evokes some sense of multi-modal physicality, or of *haptic cognition*, that is, relating to tactile perception. Perhaps this method works best by encouraging students to literally get their bodies involved when exploring the content and solving the questions. While I fully encourage directly embodied approaches of this kind, the theory appears to provide few restrictions on specific implementation details, largely owing to the amodal nature of mathematics. For example, a recent work discusses the benefits of *embodied design*, emphasizing the use of gesture and physical manipulatives (Abrahamson et al., 2020). On the other hand, that very same work also describes the use of video games, one of which was made as a level of the popular video game *LittleBigPlanet* and does not necessarily have to “get the whole body involved”; although, in fairness, they do include analysis of the verbal-gestural utterances of the players of the game. Moreover, part of what makes the embodied approach useful is that it includes a wide variety of bodies by revolving around the "barebones" of mathematical knowledge. For example, in the case of students for which visual learning is not at all an option, haptic exploration can be extremely important. In fact, solely auditory resources can theoretically be an option for people who need it, for example by using the SNARC effect to one's advantage; in practice, though, it is expected that multi-modal approaches would turn out to be empirically stronger. All this goes to say, though, that embodied design does not have to be limited to normal conceptions of "embodiment." After all, so much of the embodied theory goes to show that faculties which are often (or were previously) thought of as disembodied, such as visual perception and even abstract mathematics, do in fact have deeply embodied aspects.

In contrast, virtual games can be beneficial over real-world implementations for some purposes. A nice feature of the virtual presentation is that it allows for efficient, captivating media that supports intuitive understanding. It is becoming increasingly easy to create high-quality mathematical animations, for example with 3Blue1Brown's *manim* library (Sanderson, 2022). Furthermore, to work within the framework of embodied cognition is to recognize the intrinsic physicality of cognitive labor, but, while on the surface it may appear to backtrack on the underlying assumptions central to this work, there is in fact no contradiction in recognizing the extent to which mathematics does not preserve physical reality. Mathematics importantly involves idealization and abstraction! When one speaks of a frog hopping on lily pads, one has to work from an abstract perspective by taking each distinct lily pad to be "of the same kind," so they can be counted. There is a sort of "perfection" to the way the imaginary frogs hop on lily pads in that they must do exactly what they do and nothing else.

Nonetheless, it is clear that more research is needed to claim anything definite about this work. First, the method has not been empirically tested, so by scientific standards, it would be vacuous to claim anything about its promise as an educational strategy. This work does not outline any specific experimental design, thus that is left as an avenue of future work. Second, a possible limitation of the Frog World demo is that it is a single-player game, and these games are posited to work best when a student and a mentor can discuss and explore them together. While multiple people can tune in and play the game together, more could be done to integrate mutual exploration. It goes to show the demo level of the game is just that: a demo, and much more work could be put into the depth, richness, and playability of the game. As it stands, the game is not suitable for repeated play; future versions of the game would necessarily correct this. Last, I have claimed that these conceptual metaphor games are a general idea, but have done little in the way of providing resources for games other than Frog World. Future work would involve researching how easily other grounding metaphors and more complicated linking metaphors extend to games.

As a final criticism, while not an intrinsic limitation of the general method, the model of zooming in on one particular metaphor can be restricting for a few reasons. As explained in the literature review section, researchers have directed a fair bit of criticism towards the original work of Lakoff and Núñez which grounds this work, namely in that it posits a rather stringent view of mathematics in favor of a consistent methodology (McShane et al., 2019; Sinclair and Schiralli, 2003). I have argued that these criticisms are not earth-shattering to the application of the theory demonstrated here, but it is possible this is a hasty generalization over the vast amount of work that exists on the topic. Because a significant amount of prior literature has shown mathematical learning games to have a positive influence on learning outcomes, an empirical analysis of the effectiveness of the proposed pedagogical practice would ideally have a control game which is not grounded in the same theory. Further, not only is it possible to push the limits of one particular metaphor to the point where it becomes unhelpful, presenting multiple different contexts for the same idea could prove to be better than focusing strongly on one context. In order to understand why the generalities of mathematics are important, it can be helpful to see many particular cases in which those generalities are realized. For this reason, I encourage further research on a diverse range of metaphors.

Figure 2

An abstract frog automaton puzzle (top row) and solutions (bottom sixteen rows). The initial frog is a 3-Frog (far left, green), and the goal is to place frogs so they catch the bug (far right, red).

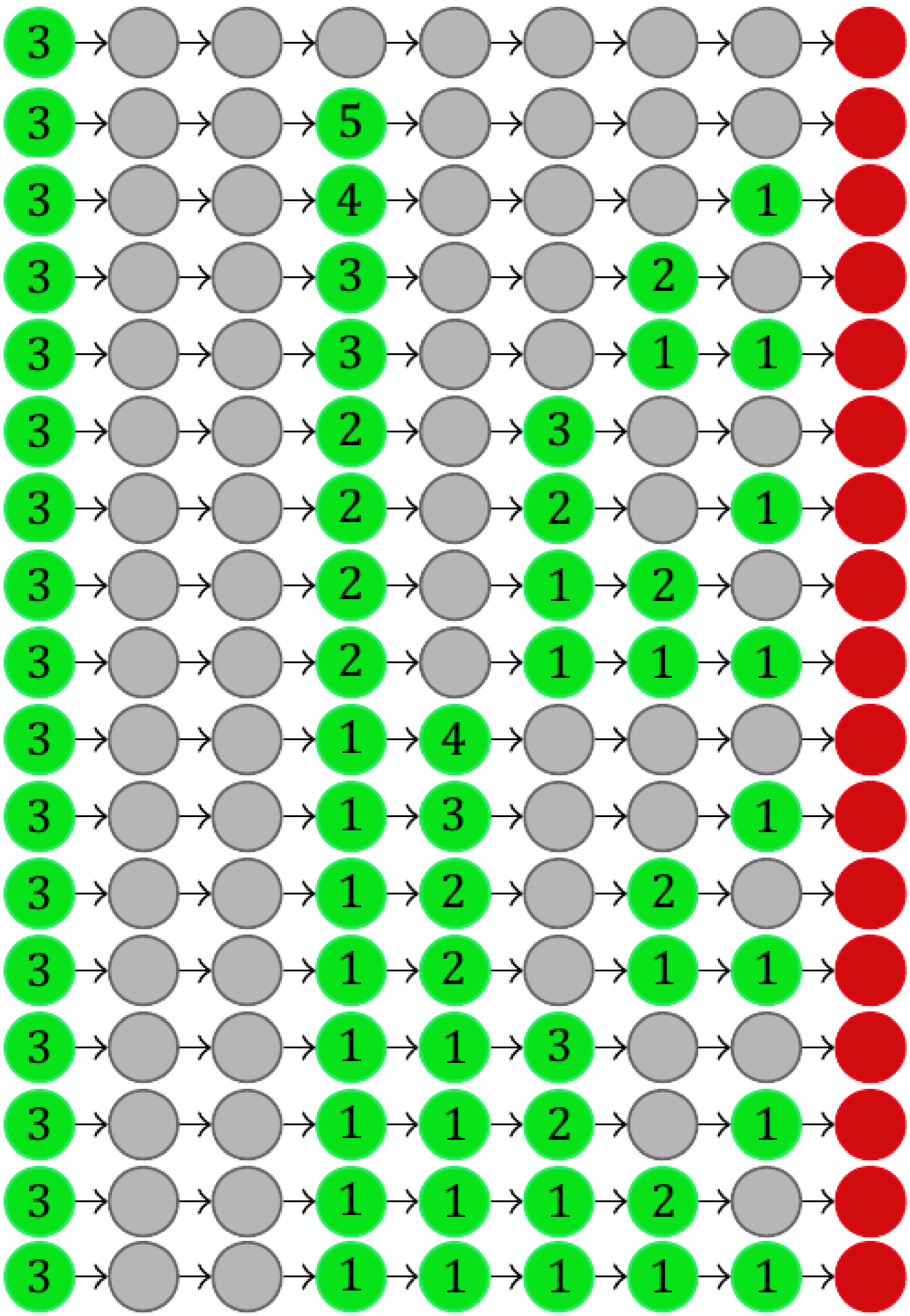


Figure 3

A frog puzzle where the frog wants to eat the fly, but the player is only allowed to use the 3-Frog and 5-Frog in either direction, and must stay on the shown lilies. There is a gap between the last lily and the one before it.

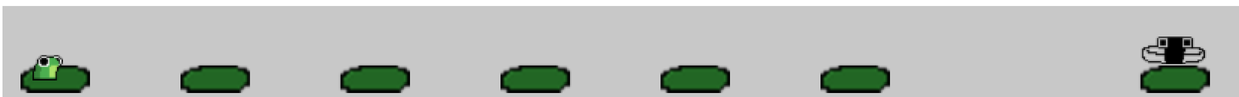
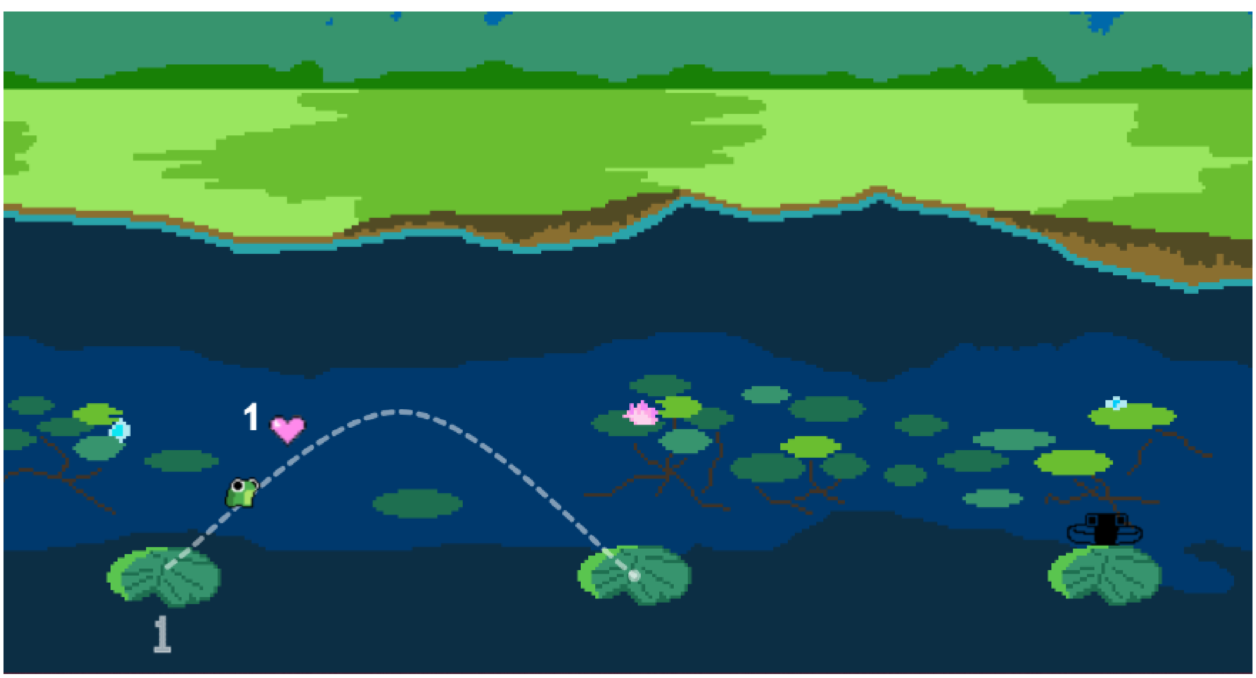


Figure 4

A screenshot from the Frog World demo. A frog hops on a lily towards a fly.



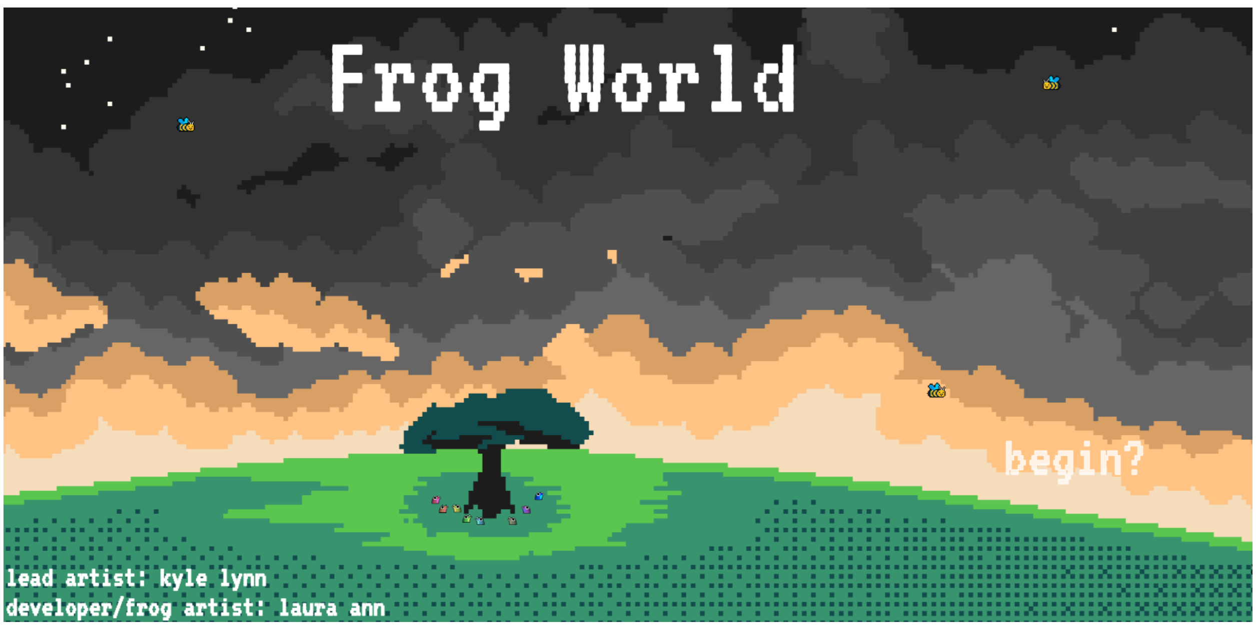
Appendix

Additional Implementation Details

As described in the main text, *Frog World* was implemented in *p5.js* ("p5.js," 2022). The demo is intended to be as playable as possible with minimal instruction, however there are some features of the game not directly addressed by either the main paper or the demo. Note that while the game is not publicly available at the time of writing this, I intend to upload the game to my website at `https://lauraann.dev`.

Figure A1

The Frog World menu. Bees fly around the screen as the camera pans around a scenic menu background with some "tree-Frogs."



Camera. All playable levels are implemented with a fully functional custom camera object. The camera object has a specified center (x, y) and a resolution given by (w, h). Each frame, objects whose bounding box intersects the rectangle $(x - w/2, y - h/2) \times (x + w/2, y + h/2)$ are drawn on the screen. The camera object is equipped with two important methods, `toWorldCoords` and `toCamCoords`; these methods convert between world space (global coordinates) and camera space (space where the origin is fixed at the camera center). In exploratory mode, pressing `Y` on the keyboard unlocks the camera and allows panning by moving the mouse to either edge of the screen. Scrolling the mouse wheel zooms in or out.

Jumping. Jumping is a fixed animation, without any simulated physics. Operations act directly on the frog's position in world space with linear interpolations of the position components of source lily and the destination lily. This formulation allows a general jump function to be used for both linear and cyclic modes. Each jump corresponds to one click, and as mentioned, jump direction is specified by the position of the mouse relative to a vertical axis. Light help lines are drawn to show where the frog will land with each jump, spatially embodying the frog's cardinality.

A secret is that pressing up and down on the keyboard changes the "type" of frog, e.g., pressing up with the 1-Frog gives the 2-Frog. However, this functionality would be more constrained in a future version of the game.

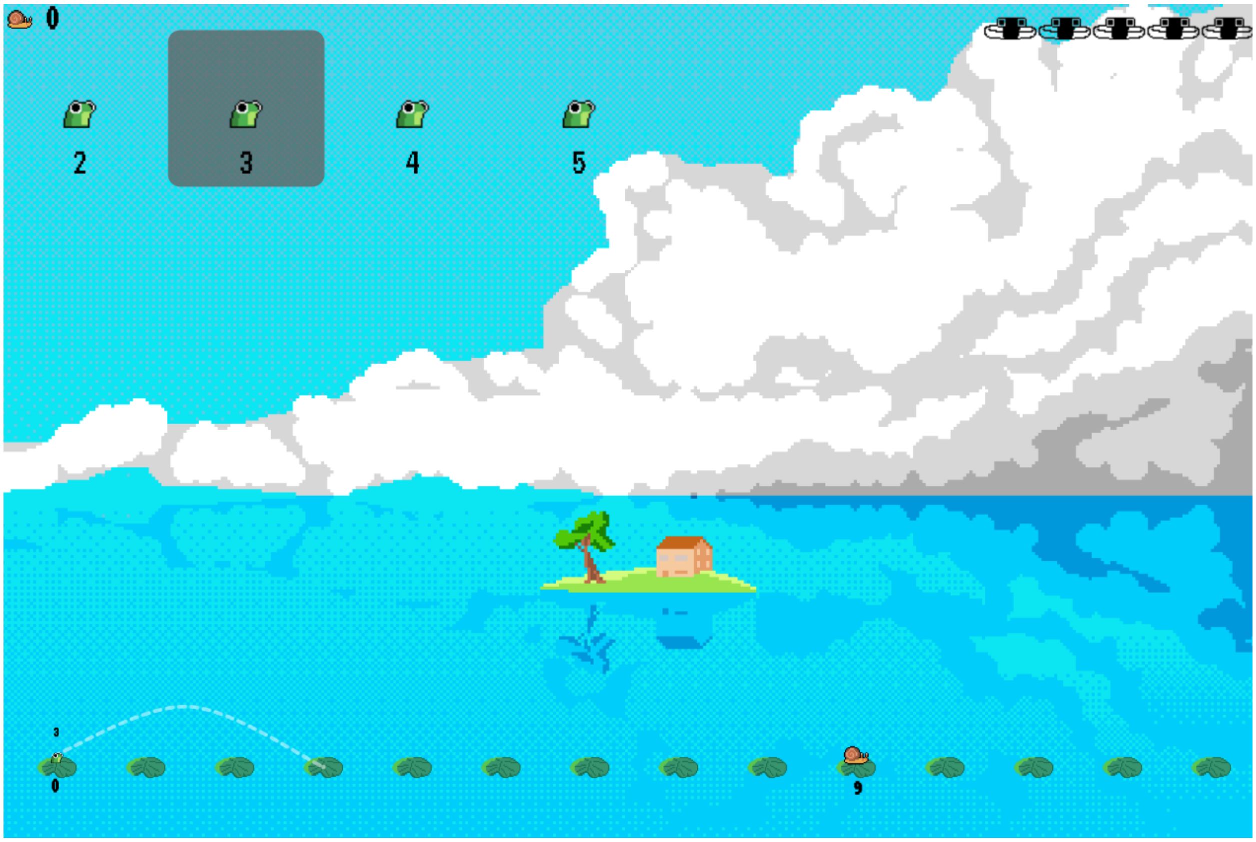
Lilies. For linear mode, the lilies are "infinitely" looped by calculating the position of the lily that intersects the left bound of the screen, and then drawing enough lilies to fill the width of the screen (depending on a given padding parameter). In cyclic mode, lily positions are generated with cosine and sine.

Objectives. The demo game has three main objectives. The first is to complete quests specified by the quest manager. The second objective is puzzles; for example, upon completing the first quest in the demo, a wizard frog "warps" the player into puzzle mode. The third objective is to sate the frogs' hunger, and it always applies. This third objective is an important means to organically encouraging solving quests in return for bugs to eat while discouraging dilly-dallying, as running out of hunger means the game is over.

Puzzles. The camera is locked and anchored in puzzle mode, and the player at first cannot move. A puzzle level resets after the player is able to see that they have picked an incorrect choice; i.e., the game lets the user try different options. An example is shown in Figure A2. In the top left, the snail counter is displayed, which increases with each level

Figure A2

A Frog World factors puzzle.



cleared. On the right is the hunger bar, which fades as the frog hops around. When the player hovers over each frog button, the help lines change to reflect the distance hopped by that frog. After the user clicks the button, they are allowed to jump with the frog to try to get the snail. If the player catches the snail, they win the level and move on; otherwise, they reset back to the beginning. In the example shown, the correct solution is to click the 3-Frog and hop three times to the right.

Art. I have rights to all art in the game. Any art that I did not directly create (i.e., anything but the frogs and flies) was the work of my friend Kyle Lynn, who contributed greatly to the aesthetics of this project.

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